ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 429 (2008) 290–292
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On the similarity of dual operators

Alexandru Tupan

Department of Mathematics, University of Wisconsin – River Falls, River Falls, WI 54022, USA

Received 15 February 2008; accepted 21 February 2008
Available online 8 April 2008
Submitted by R.A. Horn

Abstract

Let k be a field with an involution σ and $\langle\ ,\ \rangle : V \times W \to k$ a non-degenerate sesquilinear form, where V, W are n-dimensional k-spaces. Assume that $\Lambda \in \mathrm{End}(V)$ and $\Lambda^* \in \mathrm{End}(W)$ are dual operators. We show that if Λ and Λ^* are similar, then $\Lambda^* = \phi\Lambda\phi^{-1}$, where $\phi : V \to W$ is Hermitian.

AMS classification: 11E39

Keywords: Dual operator; Hermitian operator; Sesquilinear form

1. Introduction

Let k be a field with an involution $\sigma \in \mathrm{Aut}(k)$ (σ can be trivial). The action of σ will be denoted exponentially. Let $\langle\ ,\ \rangle : V \times W \to k$ be a non-degenerate sesquilinear form, where V, W are n-dimensional k-spaces. A linear map $\phi : V \to W$ is *Hermitian* if $\langle v_1, \phi v_2\rangle = \langle v_2, \phi v_1\rangle^\sigma$ for all $v_1, v_2 \in V$. Assume that $\Lambda \in \mathrm{End}(V)$ and $\Lambda^* \in \mathrm{End}(W)$ are dual operators. As usual, $k[X]$ denotes the polynomial ring in one variable. Throughout this note, all polynomials are normalized to have leading coefficient 1. Our purpose is to prove the following result.

Theorem 1. *If Λ and Λ^* are similar operators, then there exists a Hermitian operator $\phi : V \to W$ such that $\Lambda^* = \phi\Lambda\phi^{-1}$.*

An incomplete proof of Theorem 1 appeared in [1]. To be precise, the proof given in [1] starts with the simplifying assumption that the problem can be reduced to cyclic spaces, the rest of the

E-mail address: alexandru.tupan@uwrf.edu

0024-3795/$ - see front matter
doi:10.1016/j.laa.2008.02.027

paper being devoted to proving Theorem 1 for such spaces (see Lemmas 1, 2 of [1]). We will show below that the reductive assumption is actually the key part of the argument and needs to be explained carefully. Once this is done and V, W are reduced to cyclic spaces, Theorem 1 follows easily. In particular, the proofs of Lemmas 1, 2 of [1] can be replaced by some short observations.

Our result extends work from [1,2,4]. Also, it is worth noting that Theorem 1 generalizes a classical fact about complex matrices (see [3, Theorem 4.1.7]).

2. Proof of Theorem 1

We regard V, W as $k[X]$-modules, where multiplication by X is given by the action of Λ, respectively Λ^*. An operator $\phi : V \to W$ satisfying $\phi\Lambda = \Lambda^*\phi$ is the same as a $k[X]$-morphism between these modules. By duality, we have $\langle Pv, w\rangle = \langle v, P^\sigma w\rangle$ for all $(v, w) \in V \times W$ and $P \in k[X]$. Let P_Λ be the minimal polynomial of Λ. By duality again, P_Λ^σ is the minimal polynomial of Λ^*. But Λ and Λ^* are similar, so $P_\Lambda^\sigma = P_\Lambda$. The space V splits as a direct sum of modules

$$V_P = \{v \in V : P^\ell v = 0 \text{ for some } \ell > 0\},$$

where P runs through the prime factors of P_Λ. A similar fact is true for W. The construction of ϕ reduces to corresponding primary components by using the following observation. If $P \neq Q$ are prime factors of P_Λ, then $V_P \perp W_{Q^\sigma}$. Indeed, if $v \in V_P$ and $w \in W_{Q^\sigma}$, then

$$\langle v, w\rangle = \langle RP^\ell v, w\rangle + \langle Sv, (Q^\sigma)^\ell w\rangle = 0,$$

where ℓ is chosen so that $P^\ell v = (Q^\sigma)^\ell w = 0$ and $R, S \in k[X]$ satisfy $RP^\ell + SQ^\ell = 1$. Therefore we may assume that $P \in k[X]$ is prime and one of the following is true.

Case 1. $P = P^\sigma$ and V, W are P-primary.

Case 2. $P \neq P^\sigma$ and $V = V_0 \oplus \overline{V_0}$, $W = W_0 \oplus \overline{W_0}$, where V_0, W_0 are P-primary and $\overline{V_0}, \overline{W_0}$ are P^σ-primary.

In Case 1 we reduce V, W to cyclic components as follows. Choose $v \in V$ with $\mathrm{Ann}(v) = (P^\ell)$ and ℓ maximal. By non-degeneracy, there is $w \in W$ such that $\langle P^{\ell-1}v, w\rangle = 1$. Then $P^{\ell-1}w \neq 0$, so $k[X]v$ and $k[X]w$ are equal dimensional spaces. We show that

$$V = k[X]v \oplus V', \qquad W = k[X]w \oplus W', \tag{1}$$

where $V' = (k[X]w)^\perp$ and $W' = (k[X]v)^\perp$. Indeed, if $0 \neq v' \in k[X]v$, then $v' = P^\alpha Qv$, where $Q \in k[X]$ is coprime to P and $0 \leqslant \alpha < \ell$, so

$$\langle v', P^{\ell-\alpha-1}R^\sigma w\rangle = \langle P^{\ell-1}v, w\rangle \neq 0,$$

where R is an inverse of Q mod P^ℓ. Thus $k[X]v \cap V' = (0)$. The k-dimension of V' is at least $\dim_k V - \dim_k k[X]w$, so $V = k[X]v \oplus V'$. The argument for W is similar. Now we can continue inductively, replacing the pair (V, W) by (V', W').

In Case 2 we reduce V, W to cyclic components in a similar way. Choose $v \in V_0$ with $\mathrm{Ann}(v) = (P^\ell)$ and l maximal. Then choose $\overline{w} \in \overline{W_0}$ with $\langle P^{\ell-1}v, \overline{w}\rangle = 1$. Analogously, choose $\overline{v} \in \overline{V_0}$ and $w \in W_0$ with $\langle (P^\sigma)^{\ell-1}\overline{v}, w\rangle = 1$. We saw before that $V_0 \perp W_0$ and $\overline{V_0} \perp \overline{W_0}$, so as in 1 we have decompositions

$$V_0 \oplus \overline{V_0} = (k[X]v \oplus V_0') \oplus (k[X]\overline{v} \oplus \overline{V_0'}),$$
$$W_0 \oplus \overline{W_0} = (k[X]w \oplus W_0') \oplus (k[X]\overline{w} \oplus \overline{W_0'}),$$

where $V_0' = V_0 \cap (k[X]\overline{w})^{\perp}$ and $\overline{V_0'}$, W_0', $\overline{W_0'}$ are defined similarly. Continue inductively, using the same argument for $V_0' \oplus \overline{V_0'}$ and $W_0' \oplus \overline{W_0'}$.

Finally, we build ϕ for the cyclic subspaces resulting from Cases 1, 2 as follows (note that these are alternative proofs for Lemmas 1, 2 of [1]).

In Case 1, we define $\phi : k[X]v \to k[X]w$ by $\phi(X^i v) = X^i w$. The map ϕ is Hermitian if and only if $\langle X^i v, w\rangle = \langle X^i v, w\rangle^{\sigma}$ for all i. This is automatic if $\sigma = 1$. If $\sigma \neq 1$, by the duality of $k[X]v$ and $k[X]w$ we find $w' \in k[X]w$ with the property

$$\langle X^i v, w'\rangle = \begin{cases} 1 & \text{if } i = \deg P^{\ell-1}, \\ 0 & \text{if } i \neq \deg P^{\ell-1}, \quad 0 \leqslant i < \deg P^{\ell}. \end{cases}$$

Then $\langle P^{\ell-1}v, w'\rangle = 1$ and we can replace w by w' to get ϕ Hermitian.

In Case 2, we define

$$\phi : k[X]v \oplus k[X]\overline{v} \to k[X]w \oplus k[X]\overline{w}, \quad \phi(X^i v + X^j \overline{v}) = X^i w + X^j \overline{w}.$$

The map ϕ is Hermitian if and only if $\langle X^i v, \overline{w}\rangle = \langle X^i \overline{v}, w\rangle^{\sigma}$ for all i. This can be accomplished by correctly choosing w, $\overline{w}$, using the duality $k[X]v$ and $k[X]\overline{w}$, respectively $k[X]w$ and $k[X]\overline{v}$, as in Case 1. Our proof is complete.

References

[1] D.Ž. Doković, On the similarity between a linear transformation and its adjoint, Linear Algbra Appl. 6 (1973) 155–158.

[2] J.W. Duke, A note on the similarity of a matrix and its transpose conjugate, Pacific J. Math. 81 (1969) 321–323.

[3] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[4] O. Taussky, H. Zassenhaus, On the similarity transformations between a matrix and its transpose, Pacific J. Math. 9 (1959) 893–896.